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Abstract: A theorem of Bourgain states that the harmonic measure for a domain in ^{d} is supported on a set of Hausdorff dimension strictly less than d [**?**]. We apply Bourgain's method to the discrete case, i.e., to the distribution of the first entrance point of a random walk into a subset of ^{d}, $d \geq 2$. By refining the argument, we prove that for all > 0 there exists $\rho(d,) < d$ and $N(d,)$, such that for any $n > N(d,)$, any $x \in^d$, and any $A \subset 1, ..., n^d$

$$|y \in^d : \nu_{A,x}(y) \geq n^-$$

$| \leq n^{\rho(d,)}, where \nu_{A,x}(y)$ denotes the probability that y is the first entrance point of the simple random walk starting at x into A. Furthermore, ρ must converge to d as $\to \infty$.

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QUANTITATIVE ESTIMATES OF DISCRETE HARMONIC MEASURES

BY

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ABSTRACT

A theorem of Bourgain states that the harmonic measure for a domain in $\mathbb{R}^d$ is supported on a set of Hausdorff dimension strictly less than d [2]. We apply Bourgain's method to the discrete case, i.e., to the distribution of the first entrance point of a random walk into a subset of $\mathbb{Z}^d$, $d \geq 2$. By refining the argument, we prove that for all $\beta > 0$ there exists $\rho(d,\beta) < d$ and $N(d,\beta)$, such that for any $n > N(d,\beta)$, any $x \in \mathbb{Z}^d$, and any $A \subset \{1,\dots,n\}^d$

$$|\{y \in \mathbb{Z}^d \colon \nu_{A,x}(y) \geq n^{-\beta}\}| \leq n^{\rho(d,\beta)},$$

where $\nu_{A,x}(y)$ denotes the probability that y is the first entrance point of the simple random walk starting at x into A. Furthermore, ρ must converge to d as $\beta \to \infty$.

1. Introduction

Let $(S_n)_{n\in\mathbb{N}}$ be a simple random walk in $\mathbb{Z}^d$ starting at $x \in \mathbb{Z}^d$, i.e., $S_0 = x$ and

$$\mathbb{P}^x(S_{n+1} - S_n = e) = \frac{1}{2d}, \quad \|e\| = 1, \quad n \in \mathbb{N}.$$

($\|\,.\,\|$ denotes the Euclidian distance, i.e., $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$.) For $A \subset \mathbb{Z}^d$, $A \neq \emptyset$, we denote by τ_A the time of the first entrance of S into A:

$$\tau_A = \inf\{n \geq 0 \colon S_n \in A\}.$$

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The harmonic measure for A of a set $B \subset \mathbb{Z}^d$ evaluated at $x \in \mathbb{Z}^d$ is defined as

$$\omega(A, B, x) = \mathbb{P}^x(\tau_A < \infty,\ S_{\tau_A} \in B).$$

Clearly, for $x \in A$, $\omega(A, B, x) = 1\!\!1_B(x)$. For fixed $A \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, $\omega(A, \ . \ , x)$ is a measure on $\mathbb{Z}^d$ with total mass $\omega(A, \mathbb{Z}^d, x) = \omega(A, A, x) = \mathbb{P}^x(\tau_A < \infty) \in (0, 1]$. We denote by $\nu_{A,x}(y) = \omega(A, \{y\}, x)$ its density. For $x \in A^c = \mathbb{Z}^d \setminus A$, $\omega(A, B, \ . \ )$ is a harmonic function,

$$\Delta\omega(A, B, x) = \frac{1}{2d} \sum_{\|e\|=1} \omega(A, B, x + e) - \omega(A, B, x) = 0.$$

We shall prove the following theorem:

THEOREM: (A) *For all $\beta > 0$ there exists $\rho(d, \beta) < d$ and $N(d, \beta)$, such that for any $n > N(d, \beta)$, any $x \in \mathbb{Z}^d$, and any $A \subset Q^d(n) = \{1, \ldots, n\}^d$,*

$$|\{y \in \mathbb{Z}^d \colon \nu_{A,x}(y) \geq n^{-\beta}\}| \leq n^{\rho(d,\beta)}.$$

(B) *For all $\rho < d$ there exist $\beta < \infty$ and sequences $n_K \to \infty$, $x_K \in \mathbb{Z}^d$, and $A_K \subset Q^d(n_K)$ such that for all K*

$$|\{y \in \mathbb{Z}^d \colon \nu_{A_K,x_K}(y) \geq n_K^{-\beta}\}| > n_K^{\rho}.$$

Remarks: (1) If $x \in A$, the statement of Theorem (A) is trivial. Therefore we only consider $x \in A^c$. The proof of Theorem (A) is to a large extent an adaptation of Bourgain's proof [2] that the harmonic measure for a domain in $\mathbb{R}^d$ is supported on a set of Hausdorff dimension strictly less than d to the discrete case, and the proof of Theorem (B) is inspired by Jones and Makarov [5] who also treat continuous harmonic measure.

(2) The analogous theorem holds for harmonic measure conditioned on the event that A is reached, and also for harmonic measure from infinity: Let

$$\bar{\nu}_{A,x}(y) = \mathbb{P}^x(S_{\tau_A} = y |\ \tau_A < \infty),$$

and

$$\bar{\nu}_{A,\infty}(y) = \lim_{\|x\| \to \infty} \bar{\nu}_{A,x}(y).$$

(See, for example, [6], Chapter 2.1 for the existence of $\bar{\nu}_{A,\infty}$.) Then we have

(A′) For all $\beta > 0$ there exists $\rho(d, \beta) < d$ and $N(d, \beta)$, such that for any $n > N(d, \beta)$, any $x \in \mathbb{Z}^d$, and any $A \subset Q^d(n) = \{1, \ldots, n\}^d$,

$$|\{y \in \mathbb{Z}^d \colon \bar{\nu}_{A,x}(y) \geq n^{-\beta}\}| \leq n^{\rho(d,\beta)},$$

and

(A'') *For all* $\beta > 0$ *there exists* $\rho(d,\beta) < d$ *and* $N(d,\beta)$, *such that for any* $n > N(d,\beta)$ *and any* $A \subset Q^d(n) = \{1,\dots,n\}^d$,

$$|\{y \in \mathbb{Z}^d \colon \bar{\nu}_{A,\infty}(y) \geq n^{-\beta}\}| \leq n^{\rho(d,\beta)}.$$

For (A'), note first that for $d=2$, $\mathbb{P}^x(\tau_A < \infty) = 1$ for all x and A by recurrence and therefore $\bar{\nu}_{A,x} = \nu_{A,x}$. For $d \geq 3$, we have a lower bound on the hitting probability $\mathbb{P}^x(\tau_A < \infty)$ for x in a neighborhood of $Q^d(n)$,

$$\begin{aligned} \text{(1)} \qquad \mathbb{P}^x(\tau_A < \infty) &\geq \mathbb{P}^x(\tau_{\{z\}} < \infty) = \frac{G(x-z)}{G(0)} \\ &\geq \frac{c_2}{G(0)}\|x-z\|^{2-d} \geq c(a,d)n^{2-d} \end{aligned}$$

for all $z \in A$ and $x \in U^d(an) = \{-an,\dots,(a+1)n\}^d$, where G is the Green's function which satisfies (9); see Section 2.3 below. For more distant x, $\bar{\nu}_{A,x}$ doesn't change a lot any more: For $d \geq 2$, there exist constants $C_1(d)$ and $C_2(d)$ such that for all $A \subset Q^d(n)$, $y \in A$, $x \in (U^d(an))^c$ with $a \geq 2\sqrt{d}$,

$$\text{(2)} \qquad C_1\bar{\nu}_{A,x}(y) \leq \bar{\nu}_{A,\infty}(y) \leq C_2\bar{\nu}_{A,x}(y);$$

see [6], Chapter 2.1. From (1) and (2), (A') follows, and (A'') follows from (A') with (2). Similarly we have the analogs of Theorem (B).

(3) Our theorem improves a result of Benjamini [1]. In fact, it implies the following weaker statement (which is still stronger than [1]): There exists $\rho(d) < d$, such that for any $\varepsilon > 0$ there is an $N(\varepsilon)$ such that for any $n > N(\varepsilon)$, any $x \in \mathbb{Z}^d$, and any $A \subset Q^d(n) = \{1,\dots,n\}^d$, there is a set $\tilde{A} \subset A$ with

$$\omega(A,\tilde{A},x) > \omega(A,A,x) - \varepsilon \quad \text{and} \quad |\tilde{A}| < \varepsilon n^{\rho}.$$

The analogous statements hold for harmonic measure conditioned on the event that A is reached, and also for harmonic measure from infinity. Note that it is in general impossible that $\tilde{A}$ carries the full mass: Considering for example (for even n) $A = \{1,3,5,\dots,n-1\}^d$, the only set having full mass (for $x \notin A$) is A, and $|A| = (n/2)^d$.

(4) The dependence of the exponent ρ on β for 2-dimensional simple random walk paths A (the "multifractal spectrum of the harmonic measure for A") has been studied by Lawler [8]. Also for $d=2$, there is another result of Lawler [7] which gives more information on the support of harmonic measure from infinity $\bar{\nu}_{A,\infty}$ for connected sets.

2. Proof

2.1 PROOF OF THEOREM (B). Take $n_K = 2^K$. Delete from $\{1, 2, \ldots, n_K\}$ the central $\delta 2^K$ points, from the remaining two intervals of length $(1-\delta)2^{K-1}$ the central $\delta(1-\delta)2^{K-1}$ points, and so on, k $(< K)$ times. In the j-th step, we have deleted $\delta(1-\delta)^{j-1}2^{K-j+1}$ points and obtained intervals of length $(1-\delta)^j 2^{K-j}$. Let now A_K be the product of d copies of the resulting set. It consists of 2^{kd} squares of side length $(1-\delta)^k 2^{K-k}$. The total number of boundary points is

$$|\partial A_K| = 2^{kd} \cdot 2d \cdot \left[(1-\delta)^k 2^{K-k}\right]^{d-1}.$$

To estimate the harmonic measure of the points of ∂A_K we use the discrete Harnack inequality; see for example [6], Thm. 1.7.2: There exists a $c < \infty$ such that if $f\colon \mathbb{Z}^d \to [0, \infty)$ is harmonic on B_n,

$$f(x_1) \le c f(x_2), \quad \|x_1\|, \|x_2\| \le n/2, \tag{3}$$

with $B_n = \{z \in \mathbb{Z}^d\colon \|z\| < n\}$.

Consider an arbitrary point $y \in \partial A_K$, and let x_K be (for example) the central point of $Q^d(n_K)$, i.e., $x_K = (2^{K-1}, \ldots, 2^{K-1})$. $Q^d(n_K) \setminus A_K$ consists of cylinders, called j-cylinders, of width $\delta(1-\delta)^{j-1}2^{K-j+1}$, $j = 1, \ldots, k$, in one component, and of width n_K in the other components. y lies on the boundary of a j_0-cylinder for some $j_0 \le k$. Let z_0 be the point closest to y lying in the center of the j_0-cylinder. Let z_1 be the point closest to z_0 lying in the center of a (j_0-1)-cylinder. The distance from z_0 to z_1 is $\le (1-\delta)^{j_0-2}2^{K-j_0+1}$. Continue inductively to define points z_i lying closest to z_{i-1} in the center of a (j_0-i)-cylinder up to $i = j_0 - 1$. $|z_i - z_{i-1}| \le (1-\delta)^{j_0-i-1}2^{K-j_0+i}$ and $|x_K - z_{j_0-1}| \le 2^{K-1}$. Applying (3) gives

$$\begin{aligned}\nu_{A_K,x_K}(y) &\ge c^{-1/\delta}\nu_{A_K,z_{j_0-1}}(y) \ge c^{-1/\delta}c^{-2/(\delta(1-\delta))}\nu_{A_K,z_{j_0-2}}(y) \\ &\ge \ldots \ge c^{-1/\delta}\left[c^{-2/(\delta(1-\delta))}\right]^{j_0-1}\nu_{A_K,z_0}(y) \ge c^{-4k/\delta}\nu_{A_K,z_0}(y).\end{aligned}$$

We may estimate $\nu_{A_K,z_0}(y)$ simply by $\nu_{A_K,z_0}(y) \ge \tilde{c}\|z_0 - y\|^{1-d} \ge \tilde{c}2^{-K(d-1)}$ (see [6], Lemma 1.7.4). Therefore

$$\nu_{A_K,x_K}(y) \ge c^{-4k/\delta}\tilde{c}2^{-K(d-1)}.$$

Now we want $|\partial A_K| > 2^{K\rho}$ and $\nu_{A_K,x_K}(y) > 2^{-K\beta}$. This is achieved for large enough K by putting δ such that $\rho = d + 3(d-1)\log(1-\delta)/\log 2$, β such that $\beta - d + 1 = 4\log c/(\delta \log 2)$, and $k = \gamma K$ with

$$\gamma = \log\left[2(1-\delta)^{3(d-1)}\right] / \log\left[2(1-\delta)^{d-1}\right].$$

2.2 Discrete Hausdorff measure. For bounded sets $A \subset \mathbb{Z}^d$, consider coverings of A by a countable number of balls B_α in $\mathbb{Z}^d$ with center z_α and radius r_α, $A \subset \bigcup_\alpha B_\alpha$ with

$$B_\alpha = \{x \in \mathbb{Z}^d \colon \|x - z_\alpha\| \le r_\alpha\}.$$

For $0 < \rho \le d$ we define

$$h_\rho(A) = \inf\left\{ \sum_\alpha |B_\alpha|^{\rho/d}; B_\alpha \text{ ball}, A \subset \bigcup_\alpha B_\alpha \right\}.$$

Furthermore, consider a net of l-adic cubes: $\mathcal{C}_0 = \mathbb{Z}^d$, $\mathcal{C}_1 = \{$cubes $C \subset \mathbb{Z}^d$ with side length $|C|^{1/d} = l$ and lower corner $c = (k_1 l, k_2 l, \ldots, k_d l)$ with $k_i \in \mathbb{Z}\}$,

$$\mathcal{C}_j = \{C \subset \mathbb{Z}^d \colon C = \{z \in \mathbb{Z}^d \colon k_i l^j \le z_i < (k_i+1)l^j, k_i \in \mathbb{Z}, i = 1 \ldots d\}\},$$

and $\mathcal{C} = \bigcup_{j \in \mathbb{N}} \mathcal{C}_j$. Analogously to h_ρ we define

$$m_\rho(A) = \inf\left\{ \sum_\alpha |C_\alpha|^{\rho/d}; C_\alpha \in \mathcal{C}, A \subset \bigcup_\alpha C_\alpha \right\}.$$

Clearly, there exist two positive constants $t_1(d)$ and $t_2(d, l, \rho)$ such that for all $A \subset \mathbb{Z}^d$

$$h_\rho(A) \le t_1(d) m_\rho(A) \tag{4}$$

and

$$m_\rho(A) \le t_2(d, l, \rho) h_\rho(A). \tag{5}$$

By considering, for example, a ball of radius $\sqrt{l}$, one sees that the dependence of t_2 on l cannot be removed. A possible choice is

$$t_2 = 8^d l^{d-\rho}. \tag{6}$$

Analogously to Theorem 1 in Carleson [3], p. 7 (see also [9], Chapter III.4) we have the following Lemma:

Lemma 1: *There are constants t_3 and t_4, depending only on d, such that for every bounded set $A \subset \mathbb{Z}^d$ there is a discrete measure μ supported on A with*

$$\mu(B) \le t_3 |B|^{\rho/d} \quad \text{for all balls } B \subset \mathbb{Z}^d \tag{7}$$

and

$$\mu(A) \ge t_4\, h_\rho(A). \tag{8}$$

Proof: Start the construction of μ by putting $\mu_0(\{x\}) = 1$ for all $x \in A$ and $\mu_0(\{x\}) = 0$ for $x \in A^c$. Choose your favorite l and consider the cubes of $\mathcal{C}_1$. If, for some $C \in \mathcal{C}_1$, $\mu_0(C) > |C|^{\rho/d}$, reduce the density on the points of C uniformly such that $\mu_1(C) = |C|^{\rho/d}$. Continue in this way. After finitely many steps no further reduction will occur, since $\mu_k(C) \leq |A|$ for all C and k and $|A| < l^{K\rho}$ for K large enough. Put $\mu = \mu_K$.

μ satisfies

$$\mu(C) \leq |C|^{\rho/d} \quad \text{for all } C \in \mathcal{C}$$

and therefore we have (7).

From the construction of μ, each point $a \in A$ is contained in a cube C_α with $\mu(C_\alpha) = |C_\alpha|^{\rho/d}$. If there are several such cubes, choose the largest one. With this (disjoint) covering $\{C_\alpha\}$ we obtain

$$\mu(A) = \sum_\alpha \mu(C_\alpha) = \sum_\alpha |C_\alpha|^{\rho/d} \geq m_\rho(A) \geq \frac{1}{t_1(d)}\, h_\rho(A)$$

with (4). This proves (8). ∎

μ puts more mass on boundary points than on interior points. Thus it is useful for estimating the harmonic measure, which is concentrated on the boundary.

2.3 ESTIMATE OF THE TRAPPING PROBABILITY. Another useful quantity to estimate the harmonic measure in $d \geq 3$ is the Green's function G, $G(x)$ being the expected number of visits to x of the random walk starting at 0,

$$G(x) = \mathbb{E}^0\left(\sum_{j=0}^{\infty} 1\!\!1_{\{x\}}(S_j)\right) = \sum_{j=0}^{\infty} \mathbb{P}^0(S_j = x).$$

G is harmonic in $\mathbb{Z}^d \setminus \{0\}$, $\Delta G(x) = -\delta(x)$, and G has the following asymptotic behavior:

$$\lim_{\|x\| \to \infty} \frac{G(x)}{a_d \|x\|^{2-d}} = 1,$$

where $a_d = 2/((d-2)\omega_d)$, and ω_d is the volume of the unit ball in $\mathbb{R}^d$ (see for example [6], p. 31). This implies that there are constants c_1 and c_2 $(0 < c_2 < c_1)$ depending only on dimension such that we have the following upper and lower bounds,

$$G(x) \leq c_1 \|x\|^{2-d} \quad \text{and} \quad G(x) \geq c_2 \|x\|^{2-d} \quad \text{for } x \in \mathbb{Z}^d \setminus \{0\}. \tag{9}$$

In $d = 2$, G is infinite, but there exists a quantity with similar properties, namely the potential kernel

$$a(x) = \lim_{n \to \infty} \sum_{j=0}^{n} \left(\mathbb{P}^0(S_j = 0) - \mathbb{P}^0(S_j = x)\right).$$

$\Delta a(x) = \delta(x)$, and a has the following asymptotic behavior:

$$\lim_{\|x\|\to\infty}\left(a(x) - \frac{2}{\pi}\log\|x\| - k\right) = 0,$$

where k is some constant (see for example [6], p. 38). Therefore there exists a constant c such that we have the following upper and lower bounds for $x \in \mathbb{Z}^d \setminus \{0\}$,

$$a(x) \le \frac{2}{\pi}\log\|x\| + k + c \quad \text{and} \quad a(x) \ge \frac{2}{\pi}\log\|x\| + k - c. \tag{10}$$

Consider now a cube $Q \subset \mathbb{Z}^d$, and let $Q_* \subset \mathbb{Z}^d$ be a cube of size $|Q_*|^{1/d} \le q|Q|^{1/d}$, where q is a constant ($0 < q < 1$) to be determined below. Q_* is placed such that its center is as close as possible to the center of Q: If $|Q_*|^{1/d}$ and $|Q|^{1/d}$ are both even or both odd, Q and Q_* have the same center, and in the other cases, the distance of the centers is $\sqrt{d}/2$. The next lemma gives an estimate of the probability that a random walk starting in Q_* reaches a set $A \subset \mathbb{Z}^d$ before leaving Q, $\mathbb{P}^a(\tau_A < \tau_{Q^c}) = \omega(A \cup Q^c, A \cap Q, a)$:

Lemma 2: *Let $\rho \ge d-1$. Then for q small enough (depending only on d) there exists $\tilde{c}(d,q) > 0$ such that for all $a \in Q_*$,*

$$\omega(A \cup Q^c, A \cap Q, a) \ge \tilde{c}\frac{h_\rho(A \cap Q_*)}{|Q_*|^{\rho/d}}. \tag{11}$$

Proof: If $A \cap Q_* = \emptyset$, (11) holds trivially.

Let now $A \cap Q_* \ne \emptyset$ and let μ be the measure on $A \cap Q_*$ from Lemma 1. We treat first the case $d \ge 3$. Consider the function u: $\mathbb{Z}^d \to \mathbb{R}^+$,

$$u(x) = \sum_{y \in A \cap Q_*} G(x-y)\,\mu(\{y\}).$$

u is harmonic in $(A \cap Q_*)^c$. For $x \in Q_*$ and $y \in Q_*$, $\|x - y\| \le |Q_*|^{1/d}\sqrt{d}$, and therefore with (9)

$$u(x) \ge c_2 d^{(2-d)/2}|Q_*|^{(2-d)/d}\mu(A \cap Q_*) \quad \text{for } x \in Q_*. \tag{12}$$

For $x \in Q^c$ and $y \in Q_*$,

$$\|x-y\| \ge \frac{|Q|^{1/d} - |Q_*|^{1/d}}{2} \ge \frac{1-q}{2q}|Q_*|^{1/d}$$

and therefore with (9)

$$u(x) \le c_1\left(\frac{1-q}{2q}\right)^{2-d}|Q_*|^{(2-d)/d}\mu(A \cap Q_*) \quad \text{for } x \in Q^c. \tag{13}$$

Furthermore, for all $x \in \mathbb{Z}^d$,

$$u(x) \leq c_3 |Q_*|^{(2+\rho-d)/d}, \tag{14}$$

where c_3 depends only on d. This is seen as follows: First of all, with (9),

$$\sup_{x\in\mathbb{Z}^d} u(x) = \sup_{x\in B(Q_*)} u(x),$$

where $B(Q_*)$ is a ball with the same center as Q_* and radius $a/2\sqrt{d}|Q_*|^{1/d}$ with suitably chosen a ($a = 1 + 2(c_1/c_2)^{1/(d-2)}$). Now, for $x \in B(Q_*)$,

$$u(x) = \sum_{k=1}^{a\sqrt{d}|Q_*|^{1/d}} \sum_{y\in\tilde{B}_k(x)} G(x-y)\,\mu(\{y\}),$$

where $\tilde{B}_k(x) = \{y \in \mathbb{Z}^d\colon k-1 \leq \|x-y\| < k\}$. Thus

$$u(x) \leq G(0)\mu(\tilde{B}_1(x)) + \sum_{k=2}^{a\sqrt{d}|Q_*|^{1/d}} c_1(k-1)^{2-d}\mu(\tilde{B}_k(x)).$$

With $B_k(x) = \{y \in \mathbb{Z}^d\colon \|x-y\| < k\}$ we obtain

$$\begin{aligned}\sum_{k=2}^{a\sqrt{d}|Q_*|^{1/d}} (k-1)^{2-d}\mu(\tilde{B}_k(x)) &= (a\sqrt{d}|Q_*|^{1/d})^{2-d}\mu(B_{a\sqrt{d}|Q_*|^{1/d}}(x)) - \mu(B_1(x))\\ &\quad + \sum_{k=2}^{a\sqrt{d}|Q_*|^{1/d}} \left((k-1)^{2-d} - k^{2-d}\right)\mu(B_k(x)).\end{aligned}$$

From (7) we have $\mu(B_k(x)) \leq \tilde{t}_3 k^\rho$ for a suitable $\tilde{t}_3$ depending only on d. Then

$$\begin{aligned}\sum_{k=2}^{a\sqrt{d}|Q_*|^{1/d}} \left((k-1)^{2-d} - k^{2-d}\right)\mu(B_k(x)) &\leq t_3' \sum_{k=2}^{a\sqrt{d}|Q_*|^{1/d}} k^{1-d+\rho}\\ &\leq t_3' \int_0^{a\sqrt{d}|Q_*|^{1/d}+1} x^{1-d+\rho}dx\\ &= \frac{t_3'}{2-d+\rho}(a\sqrt{d}|Q_*|^{1/d}+1)^{2-d+\rho}\\ &\leq t_3'(a\sqrt{d}|Q_*|^{1/d}+1)^{2-d+\rho}\end{aligned}$$

for $\rho \geq d-1$, where t_3' depends only on d. Putting everything together, we obtain (14).

Consider now

$$\bar{u}(x) = \frac{1}{\sup_{y\in\mathbb{Z}^d} u(y)}\Big(u(x) - \sup_{y\in Q^c} u(y)\Big).$$

$\bar{u}(x) \le 1$ for all $x \in \mathbb{Z}^d$, and $\bar{u}(x) \le 0$ for $x \in Q^c$. Compare $\bar{u}(x)$ with $\omega(A\cup Q^c, A\cap Q, x)$: Application of the maximum principle (see for example [6], p. 25) to $\bar{u}-\omega$ on $A^c\cap Q$ yields $\bar{u}\le\omega$ there, and on $A\cap Q$ we have $\omega = 1 \ge \bar{u}$. Therefore

$$\omega(A\cup Q^c, A\cap Q, x) \ge \bar{u}(x) \quad \text{for all } x\in Q.$$

Together with (12), (13), (14), and (8), we obtain for $a \in Q_*$

$$\begin{aligned}\omega(A\cup Q^c, A\cap Q, a) &\ge \frac{\mu(A\cap Q_*)}{c_3|Q_*|^{(2+\rho-d)/d}}\left(c_2 d^{(2-d)/2} - c_1\left(\frac{1-q}{2q}\right)^{2-d}\right)|Q_*|^{(2-d)/d}\\ &\ge \tilde{c}\,\frac{h_\rho(A\cap Q_*)}{|Q_*|^{\rho/d}}\end{aligned}$$

if we choose q so small that $c_2 d^{(2-d)/2} - c_1\left((1-q)/2q\right)^{2-d}$ is positive. This proves Lemma 2 in the case $d\ge 3$.

For $d=2$, the analogous construction using instead of the Green's function G the potential kernel a with the estimates (10) does the job. ∎

Choose now q so that Lemma 2 holds.

2.4 An alternative for the cubes of $\mathcal{C}$. The estimate of the trapping probability (11) leads to an alternative for the cubes C of $\mathcal{C}$: Either we have a local estimate of the Hausdorff measure of $A\cap C$ or the harmonic measure is localized on the outer shells of C. Cubes of the first kind will be called (H)-cubes, those of the second kind (L)-cubes.

Consider now some $A \subset Q^d(n)$ and some $x \in \mathbb{Z}^d$. We abbreviate $\omega(B) = \omega(A, B, x)$. For $C \in \mathcal{C}_j$, $x \in (A\cup C)^c$, define (see Fig. 1)

$$\begin{aligned} C_1 &= C \setminus \text{outer subcubes } Q\in\mathcal{C}_{j-1}, Q\subset C,\\ C_2 &= C_1 \setminus \text{outer } Q\text{'s in } C_1,\\ \ldots\quad C_{\bar{l}} &= C_{\bar{l}-1}\setminus \text{outer } Q\text{'s in } C_{\bar{l}-1},\end{aligned}$$

with $\bar{l} = l/6$. For $x \in C\setminus A$, define the C_k by successively removing layers of Q-cubes around the cube Q with $x\in Q$, and, if the boundary of C is reached, remove also successively layers of outer cubes like above.

LEMMA 3: *Let $\delta > 0$ be small enough. Then for all l there exists $\rho < d$ such that each cube $C \in \mathcal{C}_j$, $j \geq 2$, satisfies one of the following conditions:*

$$\text{(H)} \quad m_\rho(A \cap C) < |C|^{\rho/d},$$

$$\text{(L)} \quad \omega(C_{\bar{l}}) \leq \frac{(1 - c_4\delta)^{\bar{l}-1}}{c_4\delta}\, \omega(C),$$

where c_4 is some constant depending only on d, $0 < c_4 < 1$.

Proof: Let $Q \in \mathcal{C}_{j-1}$ be a subcube of C, and let Q_* be the cube of size $|Q_*|^{1/d} = \left[q|Q|^{1/d}\right]$ in the middle of Q. From Lemma 2, one of the following alternatives holds:

$$\omega(A \cup Q^c, A \cap Q, a) \geq \delta \quad \text{for all } a \in Q_*, \tag{15}$$

$$h_\rho(A \cap Q_*) < \frac{\delta}{\tilde{c}}\, |Q_*|^{\rho/d}. \tag{16}$$

We shall show that if (15) holds for all subcubes $Q \subset C$, i.e., if we have a lower bound for the trapping probability, then (L) holds for C, because the harmonic measure will be concentrated on the outer shells. On the other hand, if there is one subcube Q with (16), we can estimate m_ρ of $A \cap C$.

FIRST CASE: There is a subcube $Q \subset C$, $Q \in \mathcal{C}_{j-1}$, satisfying (16). Then with (5),

$$m_\rho(A \cap Q_*) < \frac{t_2(d,l,\rho)\,\delta}{\tilde{c}}\, |Q_*|^{\rho/d},$$

and

$$\begin{aligned} m_\rho(A \cap C) &\leq m_\rho(C \setminus Q) + m_\rho(Q \setminus Q_*) + m_\rho(A \cap Q_*) \\ &\leq (l^d - 1)l^{(j-1)\rho} + l^d(1 - q/2)^d l^{(j-2)\rho} + \frac{t_2\,\delta}{\tilde{c}} q^\rho l^{(j-1)\rho}. \end{aligned}$$

Now (H) follows if

$$l^d - 1 + l^{d-\rho}(1 - q/2)^d + \frac{t_2(d,l,\rho)\,\delta}{\tilde{c}} q^\rho < l^\rho. \tag{17}$$

Plug in (6) and choose δ so small that (17) for $\rho = d$ is satisfied, i.e., such that $(1 - q/2)^d + 8^d\delta q^d/\tilde{c} < 1$. Then for all l there exists $\rho < d$ such that (17) still holds. Note that for large l and small $d - \rho$, (17) leads to

$$d - \rho \approx \frac{b}{l^d \log l} \tag{18}$$

with $b = 1 - [(1 - q/2)^d + 8^d\delta q^d/\tilde{c}]$. We shall later choose l very large and increasing with β. Thus our $d - \rho$ goes to 0 as $\beta \to \infty$.

SECOND CASE: All subcubes $Q \subset C$, $Q \in \mathcal{C}_{j-1}$, satisfy (15). Since the probability of running into A before leaving Q is everywhere high, it is hard for the random walk to enter much into the cube before having run into A, i.e., the harmonic measure of the cubes deep inside C will be very small. From the strong Markov property (see for example [6], Theorem 1.3.2) we obtain

$$\begin{aligned}\omega(A\cup C_k, C_k, x) &= \mathbb{P}^x(\tau_{A\cup C_k}<\infty,\ S_{\tau_{A\cup C_k}}\in C_k)\\ &= \sum_{y\in\partial C_{k-1}} \mathbb{P}^y(\tau_{A\cup C_k}<\infty,\ S_{\tau_{A\cup C_k}}\in C_k)\\ &\qquad\qquad \cdot\mathbb{P}^x(\tau_{A\cup C_{k-1}}<\infty,\ S_{\tau_{A\cup C_{k-1}}}=y)\\ &\le \sup_{y\in\partial C_{k-1}} \omega(A\cup C_k, C_k, y)\ \omega(A\cup C_{k-1}, C_{k-1}, x).\end{aligned}$$

(Here, $\partial A=\{x\in A\colon \exists y\in A^c \text{ with } \|x-y\|=1\}$.) Iterating this estimate, we get

$$\omega(C_{\bar l})\le \omega(A\cup C_{\bar l}, C_{\bar l}, x)\le \omega(A\cup C_1, C_1, x)\prod_{k=2}^{\bar l}\sup_{y\in\partial C_{k-1}}\omega(A\cup C_k, C_k, y). \tag{19}$$

On the other hand, using $\tau_{A\cup C_1}\le\tau_A$ and the strong Markov property,

$$\begin{aligned}\omega(C) &\ge \sum_{y\in\partial C_1}\mathbb{P}^x(\tau_A<\infty,\ S_{\tau_A}\in A\cap C, S_{\tau_{A\cup C_1}}=y)\\ &= \sum_{y\in\partial C_1}\mathbb{P}^y(\tau_A<\infty,\ S_{\tau_A}\in A\cap C)\mathbb{P}^x(\tau_{A\cup C_1}<\infty,\ S_{\tau_{A\cup C_1}}=y)\\ &\ge \inf_{y\in\partial C_1}\omega(A, A\cap C, y)\ \omega(A\cup C_1, C_1, x). \qquad (20)\end{aligned}$$

We shall show below that there exists a constant $c_4(d,q)$ such that

$$\omega(A, A\cap C, y)\ge c_4\delta \quad \text{for all } y\in\partial C_1, \tag{21}$$

and for $k=2,\dots,\bar l$

$$\omega(A\cup C_k, C_k, y)\le 1-c_4\delta \quad \text{for all } y\in\partial C_{k-1}. \tag{22}$$

These estimates, together with (19) and (20), yield (L).

It remains to prove (21) and (22): Let $y\in\partial C_1$. Consider the cube $\tilde Q$ formed from 2^d subcubes $Q\in\mathcal{C}_{j-1}$ of C "around" y, i.e., the side length of $\tilde Q$ is $2l^{j-1}$, and the distance of y from the center of $\tilde Q$ is $\le l^{j-1}/2+1$ (see Fig. 1). We have $\tilde Q\subset C$, $\tilde Q\cap C_2=\emptyset$. Enumerate the Q-cubes in $\tilde Q$:

$$\tilde Q=\bigcup_{k=1}^{2^d} Q^{(k)},$$

and let

$$\tilde{Q}_* = \bigcup_{k=1}^{2^d} Q_*^{(k)}.$$

Then, using again the strong Markov property,

$$\begin{aligned}
\omega(A, A\cap C, y) &= \mathbb{P}^y(\tau_A < \infty, S_{\tau_A} \in A\cap C) \\
&\geq \mathbb{P}^y(\tau_{\tilde{Q}_*} < \tau_{\tilde{Q}^c} \leq \infty, \exists t \in [\tau_{\tilde{Q}_*}, \tau_{\tilde{Q}^c}) \text{ with } S_t \in A) \\
&= \sum_{a\in\tilde{Q}_*} \mathbb{P}^a(\tau_A < \tau_{\tilde{Q}^c})\, \mathbb{P}^y(\tau_{\tilde{Q}_*\cup\tilde{Q}^c} < \infty, S_{\tau_{\tilde{Q}_*\cup\tilde{Q}^c}} = a) \\
&\geq \sum_{k=1}^{2^d} \sum_{a\in\tilde{Q}_*^{(k)}} \mathbb{P}^a(\tau_A < \tau_{Q^{(k)c}})\, \mathbb{P}^y(\tau_{\tilde{Q}_*\cup\tilde{Q}^c} < \infty, S_{\tau_{\tilde{Q}_*\cup\tilde{Q}^c}} = a) \\
&\geq \delta\, \omega(\tilde{Q}_*\cup\tilde{Q}^c, \tilde{Q}_*, y),
\end{aligned}$$

where we have used that all subcubes $Q \subset C$, $Q \in \mathcal{C}_{j-1}$, satisfy (15).

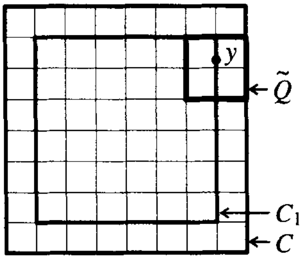

Figure 1. For $d = 2$ and $l = 8$, this is a sketch of a cube $C \in \mathcal{C}_j$ (for some j) together with its subcubes of $\mathcal{C}_{j-1}$. By removing the outer layer of subcubes, one obtains C_1. For $y \in \partial C_1$, $\tilde{Q}$ is the union of the 4 nearest subcubes.

To see that there exists c_4, independent of l and j, with $\omega(\tilde{Q}_*\cup\tilde{Q}^c, \tilde{Q}_*, y) \geq c_4$, remember that as a function of y, $\omega(\tilde{Q}_*\cup\tilde{Q}^c, \tilde{Q}_*, y)$ is (lattice) harmonic on $\tilde{Q}_*^c\cap\tilde{Q}$ with boundary values $\omega = 1$ on $\tilde{Q}_*$ and $\omega = 0$ on $\tilde{Q}^c$. Hence, the scaled function $\bar{\omega}_m(x) = \omega(\tilde{Q}_*\cup\tilde{Q}^c, \tilde{Q}_*, mx + z)$ with $m = 2l^{j-1} + 1$, $2l^{j-1}$ the side length of $\tilde{Q}$, and suitable shift z, converges as $m \to \infty$ to the unique solution of $\Delta f = 0$ on B, $f(x) = 0$ on the outer boundary of B and $f(x) = 1$ on the inner boundaries of B, where B is the "limit" of the scaled domains $m^{-1}(\tilde{Q}_*^c \cap \tilde{Q} - z)$ as $m \to \infty$; see Fig. 2. Since the convergence is uniform on compact subsets of B [4], we have a lower bound c_4 for $\omega(\tilde{Q}_*\cup\tilde{Q}^c, \tilde{Q}_*, y)$ for all l, j, and all $y = mx + z$ with x in a region S around the middle halves of the middle axes of B (see Fig. 2). This proves (21).

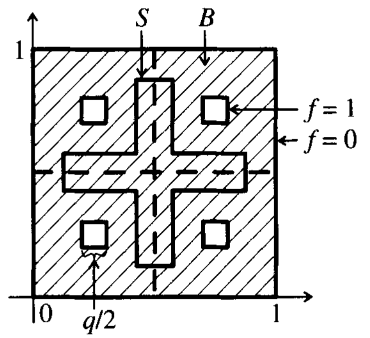

Figure 2. For $d = 2$, this is a sketch of the domain B (hatched) $= (0,1)^2 \setminus$ the 4 little squares of side length $q/2$. B corresponds to $\tilde{Q} \setminus \tilde{Q}_*$, i.e., $(mB + z) \cap \mathbb{Z}^d$, for suitable scale m and shift z, equals $\tilde{Q} \setminus \tilde{Q}_*$. The dashed middle axis lines correspond to the boundaries of the subcubes making up $\tilde{Q}$. The region S is a neighborhood of the points $x = m^{-1}(y - z)$ for those y's which are possible for $\tilde{Q}$, i.e., points on the middle half of a middle axis. The harmonic function f on B with boundary values $f = 0$ on the outer boundary of B and $f = 1$ on the boundaries of the inner squares is bounded away from 0 on S.

The proof of (22) is analogous: for $y \in \partial C_{k-1}$, put $\tilde{Q}$ to be the cube consisting of 2^d subcubes of C "around" y. Then $\tilde{Q} \cap C_k = \emptyset$ and $\tilde{Q} \subset C$. Thus

$$\begin{aligned} \omega(A \cup C_k, C_k, y) &= \mathbb{P}^y(\tau_{A \cup C_k} < \infty, S_{\tau_{A \cup C_k}} \in C_k) \\ &= \mathbb{P}^y(\tau_{A \cup C_k} < \infty) - \mathbb{P}^y(\tau_{A \cup C_k} < \infty, S_{\tau_{A \cup C_k}} \in A \setminus C_k) \\ &\leq 1 - \mathbb{P}^y(\tau_{\tilde{Q}_*} < \tau_{\tilde{Q}^c} \leq \infty, \exists t \in [\tau_{\tilde{Q}_*}, \tau_{\tilde{Q}^c}) \text{ with } S_t \in A) \\ &\leq 1 - c_4 \delta, \end{aligned}$$

with the same argument as above. ∎

2.5 Proof of Theorem (A). Let now $\beta > 0$ and $n > N(\beta)$ (to be chosen below). Let $A \subset Q^d(n)$, $x \in \mathbb{Z}^d$, and let $k^* \in \mathbb{N}$ be such that $l^{k^*} \geq n > l^{k^*-1}$. To the lower bound $N(\beta)$ there will correspond a K^* such that $N(\beta) = l^{K^*}$. We construct Bourgain's tree $\mathcal{T}$: starting with $C_0 = \{1, \ldots, l^{k^*}\}^d \in \mathcal{C}_{k^*}$, we associate to each (L)-cube $C \in \mathcal{C}_j$ its l^d subcubes in $\mathcal{C}_{j-1}$, and to each (H)-cube we associate a family $\{C_\alpha\}$ with $C_\alpha \subset C$, $A \cap C \subset \bigcup_\alpha C_\alpha$, and $\sum_\alpha |C_\alpha|^{\rho/d} < |C|^{\rho/d}$ (which exists according to Lemma 3). The elements of the tree are labeled by complexes $\gamma = (\gamma_1, \ldots, \gamma_k)$: C_0 has the label $\gamma = (\gamma_1) = (0)$, its descendants have the label $\gamma = (\gamma_1, \gamma_2) = (0, \gamma_2)$, and so on.

We stop the decomposition when the cube is in $\mathcal{C}_1$ or $\mathcal{C}_0$ (because then Lemma 3 doesn't apply any more). Thus each branch is at most k^* long. Denote by $\gamma|k$ the restriction of γ to the first k digits. If $\tilde{k}$ is the length of γ, we call $C_{\gamma|1}, C_{\gamma|2}, \ldots, C_{\gamma|\tilde{k}-1}$ the "ancestors" of C_γ. Let $\mathcal{T}^*$ denote the set of the labels of the final cubes. We have

$$A \subset \bigcup_{\gamma \in \mathcal{T}^*} C_\gamma. \tag{23}$$

Given a maximal element $\gamma \in \mathcal{T}^*$ of length $\tilde{k}$, we denote by τ_k the length of the label of the k-th (L)-cube appearing in the sequence $C_{\gamma|1}, C_{\gamma|2}, \ldots$ of ancestors of C_γ, i.e., $C_{\gamma|\tau_k}$ is the k-th (L)-cube, and $\tau_1 < \tau_2 < \ldots < \tilde{k}$. ($\tau_k = \infty$ and $\gamma|\tau_k = \gamma$ if there are less than k (L)-cubes in the sequence $C_{\gamma|1}, C_{\gamma|2}, \ldots$ of ancestors of C_γ.)

(a) **Inner cubes.** The subcubes $C_{\gamma|\tau_k+1}$ of an (L)-cube $C_{\gamma|\tau_k}$ are distinguished according to whether they lie in $(C_{\gamma|\tau_k})_{\bar{l}}$ or not. If $x \in (A \cup C)^c$, the number of subcubes which lie in $(C_{\gamma|\tau_k})_{\bar{l}}$ is $(l-2\bar{l})^d = (2/3)^d l^d$, and if $x \in C \setminus A$, the number of subcubes which lie in $(C_{\gamma|\tau_k})_{\bar{l}}$ is simply estimated as $\geq (l-2\bar{l})^d - (2\bar{l}+1)^d \geq pl^d$ with $p = (2/3)^d - (1/2)^d$. To have a fixed proportion of "inner" subcubes (this simplifies the argument in part (c) below), we shall choose for any (L)-cube pl^d subcubes from those subcubes $C_{\gamma|\tau_k+1} \subset (C_{\gamma|\tau_k})_{\bar{l}}$ to call them "inner" subcubes.

Let $k_1 = k^*/3$ and $k_2 = (p/2)k_1$. Let

$$\begin{aligned} \mathcal{T}_< &= \{\gamma \in \mathcal{T}^* \colon \tau_{k_1}(\gamma) = \infty\}, \\ \mathcal{T}_i &= \{\gamma \in \mathcal{T}^* \colon \tau_{k_1}(\gamma) < \infty, \\ &\qquad \text{at least } k_2 \text{ of } C_{\gamma|\tau_1+1}, C_{\gamma|\tau_2+1}, \ldots, C_{\gamma|\tau_{k_1}+1} \text{ are inner}\}, \end{aligned}$$

and $\mathcal{T}_o = \mathcal{T}^* \setminus (\mathcal{T}_< \cup \mathcal{T}_i)$. If $C_{\gamma|\tau_k+1}$ is inner, we have from Lemma 3

$$\omega(C_{\gamma|\tau_k+1}) \leq \omega\left((C_{\gamma|\tau_k})_{\bar{l}}\right) \leq \frac{(1-c_4\delta)^{\bar{l}-1}}{c_4\delta}\, \omega(C_{\gamma|\tau_k}),$$

and if not, then in any case

$$\omega(C_{\gamma|\tau_k+1}) \leq \omega(C_{\gamma|\tau_k}).$$

Then for $y \in \bigcup_{\gamma \in \mathcal{T}_i} C_\gamma$ we have (with γ such that $y \in C_\gamma$)

$$\begin{aligned}\nu_{A,x}(y) \leq \omega(C_\gamma) \leq \omega(C_{\gamma|\tau_{k_1}+1}) &\leq \left(\frac{(1-c_4\delta)^{\bar{l}-1}}{c_4\delta}\right)^{k_2} \omega(C_{\gamma|\tau_1}) \\ &\leq \left(\frac{(1-c_4\delta)^{\bar{l}-1}}{c_4\delta}\right)^{k_2}.\end{aligned}$$

Now choose l such that

$$\left(\frac{(1-c_4\delta)^{\bar{l}-1}}{c_4\delta}\right)^{k_2} < l^{-k^*\beta},$$

i.e.,

$$\frac{p}{6}\left(\frac{l}{6}-1\right)\log\frac{1}{1-c_4\delta} - \frac{p}{6}\log\frac{1}{c_4\delta} > \beta \log l.$$

Then

$$\bigcup_{\gamma \in \mathcal{T}_i} C_\gamma \subset \{y \in \mathbb{Z}^d \colon \nu_{A,x}(y) < n^{-\beta}\}.$$

With (23) we obtain

$$\{y \in \mathbb{Z}^d \colon \nu_{A,x}(y) \geq n^{-\beta}\} \subset \bigcup_{\gamma \in \mathcal{T}_< \cup \mathcal{T}_\circ} C_\gamma.$$

We shall show that $\sum_{\gamma \in \mathcal{T}_<} |C_\gamma| \leq n^{-\tilde{\rho}}/2$ and $\sum_{\gamma \in \mathcal{T}_\circ} |C_\gamma| \leq n^{-\tilde{\rho}}/2$ with $\tilde{\rho} = (\rho + d)/2$, where $\rho < d$ comes from Lemma 3. This then proves Theorem (A).

(b) **Estimate of** $\sum_{\gamma \in \mathcal{T}_<} |C_\gamma|$. If C_γ is of type (H), then

$$\sum_{\gamma_k = 1,\ldots;(\gamma,\gamma_k)\in\mathcal{T}} |C_{(\gamma,\gamma_k)}|^{\rho/d} \leq |C_\gamma|^{\rho/d},$$

and if $C_\gamma \in \mathcal{C}_j$ is of type (L), then we have

$$\sum_{\gamma_k=1,\ldots,l^d} |C_{(\gamma,\gamma_k)}|^{\rho/d} = l^d l^{(j-1)\rho} = l^{d-\rho}|C_\gamma|^{\rho/d}.$$

Thus

$$\begin{aligned}
|C_0|^{\rho/d} &\geq \sum_{\gamma|\tau_1(\gamma);\gamma\in\mathcal{T}_<} |C_{\gamma|\tau_1(\gamma)}|^{\rho/d} \\
&\geq l^{-(d-\rho)} \sum_{\gamma|\tau_1(\gamma)+1;\gamma\in\mathcal{T}_<} |C_{\gamma|\tau_1(\gamma)+1}|^{\rho/d} \\
&\geq l^{-(d-\rho)} \sum_{\gamma|\tau_2(\gamma);\gamma\in\mathcal{T}_<} |C_{\gamma|\tau_2(\gamma)}|^{\rho/d} \\
&\geq l^{-2(d-\rho)} \sum_{\gamma|\tau_2(\gamma)+1;\gamma\in\mathcal{T}_<} |C_{\gamma|\tau_2(\gamma)+1}|^{\rho/d} \\
\cdots &\geq l^{-(k_1-1)(d-\rho)} \sum_{\gamma\in\mathcal{T}_<} |C_\gamma|^{\rho/d}
\end{aligned}$$

and therefore

$$\sum_{\gamma\in\mathcal{T}_<} |C_\gamma| = \sum_{\gamma\in\mathcal{T}_<} |C_\gamma|^{\rho/d}|C_\gamma|^{(d-\rho)/d} \leq l^{k_1(d-\rho)}l^{k^*\rho}.$$

For our choice of k_1 and $\tilde{\rho}$ we have indeed

$$l^{k_1(d-\rho)}l^{k^*\rho} \leq \frac{1}{2}\,l^{(k^*-1)\tilde{\rho}} \leq \frac{1}{2}\,n^{\tilde{\rho}}$$

for k^* larger than some K^*.

(c) **Estimate of** $\sum_{\gamma\in\mathcal{T}_o}|C_\gamma|$. Remember that $\mathcal{T}_o = \{\gamma\in\mathcal{T}^*\colon \tau_{k_1}(\gamma)<\infty$, less than k_2 of $C_{\gamma|\tau_1+1}, C_{\gamma|\tau_2+1},\ldots,C_{\gamma|\tau_{k_1}+1}$ are inner$\}$. It is easy to see that

$$\sum_{\substack{\gamma\in\mathcal{T}^*\colon\, \tau_{k_1}<\infty,\\ k \text{ of } C_{\gamma|\tau_1+1}, C_{\gamma|\tau_2+1},\\ \ldots, C_{\gamma|\tau_{k_1}+1} \text{are inner}}} |C_\gamma| \leq b(k;k_1,p)|C_0| = \binom{k_1}{k}p^k(1-p)^{k_1-k}|C_0|,$$

$b(k;k_1,p)$ being the binomial distribution, i.e., the distribution of $\sum_{i=1}^{k_1}X_i$, where the X_i are independent $\{0,1\}$-valued random variables with $P(X_i=1)=p$ for all i. For $0<a<p$, we have from application of Markov's inequality to $\exp(\xi\sum_{i=1}^{k_1}X_i)$

$$P\left(\sum_{i=1}^{k_1}X_i \leq ak_1\right) \leq e^{-k_1 I_p(a)}$$

with

$$I_p(a) = a\log\frac{a}{p} + (1-a)\log\frac{1-a}{1-p}.$$

(This is an elementary case of Cramér's theorem.) With $a = k_2/k_1 = p/2$, $I_p(a)$ depends only on d. Then

$$\sum_{\gamma \in \mathcal{T}_o} |C_\gamma| \leq \sum_{j=0}^{k_2-1} b(j; k_1, p)|C_0| \leq e^{-k_1 I} l^{k^* d},$$

and with our choice of the constants, noting also (18),

$$e^{-k_1 I} l^{k^* d} \leq \frac{1}{2} l^{(k^*-1)\tilde{\rho}} \leq \frac{1}{2} n^{\tilde{\rho}},$$

for k^* larger than some K^*. ∎

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